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LETTER TO THE EDITOR

The Painlevé property, Lax pair, auto-Bäcklund transformation and recursion operator of a perturbed Korteweg–de Vries equation

B V Baby

International Centre for Theoretical Physics, Miramare, 34100, Trieste, Italy

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Abstract. Using Painlevé property analysis we obtained the Lax pair, auto-Bäcklund transformation and recursion operator of a perturbed Korteweg-de Vries equation, $u_t + 6uu_x + u_{xxx} + f(t)u = 0$. The exact form of $f(t)$ for which the system is completely integrable is reported in this letter.

The unidirectional propagation of a solitary wave in a canal of uneven bottom is described by the perturbed Korteweg-de Vries (PKdV) equation (Johnson 1973, Kakutani 1971, Knickerbocker and Newell 1980, 1985)

$$u_{,t} + 6uu_{,x} + u_{,xxx} + f(t)u = 0. \tag{1}$$

In this letter we study the Painlevé property (PP) of equation (1). The Lax pair obtained from PP analysis is used for constructing the recursion operator, the existence of an infinite set of commuting symmetries is shown by using Olver's condition and the possible form of $f(t)$ for the integrability of the system (equation (1)) is found.

As in the method described by Weiss *et al* (1983), we consider the Lorentz series expansion of $u(x, t)$, given by

$$u(x,t) = \phi^{\alpha}(x,t) \sum_{j=0}^{\infty} u_j(x,t)\phi^{j}(x,t) \tag{2}$$

where $u_j(x, t)$ and $\phi(x, t)$ are analytic functions in a neighbourhood of the manifold

$$\phi(x,t) = 0 \tag{3}$$

and α is an integer to be determined. Substituting (2) in (1), a leading-order term analysis uniquely determines α to be -2. The recursion relation for $u_j(x, t)$ is given by

$$\begin{aligned} u_{j-3,t} &+ (j-4)u_{j-2}\phi_{,t} + 6\sum_{k=0}^{j} u_{j-k}(u_{k-1,x} + (k-2)u_k\phi_k) + f(t)u_{j-3} + u_{j-3,xx} \\ &+ 3(j-4)u_{j-2,xx}\phi_{,x} + 3(j-3)(j-4)u_{j-1,x}\phi_{,x}^2 \\ &+ 3(j-4)u_{j-2,x}\phi_{,xx} + (j-2)(j-3)(j-4)u_j\phi_{,x}^3 \\ &+ 3(j-3)(j-4)u_{j-1}\phi_{,x}\phi_{,xx} + (j-4)u_{j-2}\phi_{,xx} = 0 \end{aligned} \tag{4}$$

where $\phi_{,x} = \partial\phi/\partial x$, $u_{j,x} = \partial u_j/\partial x$, etc. Collecting the terms involving u_j, it is readily found that $u_j(x, t)$ are not defined for $j = -1, 4, 6$. These values of j are called

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resonances. Since $j=-1$ is not admissible, the possible resonances are $j=4, 6$ and, corresponding to these values of j, the $u_j(x, t)$ are arbitrary.

Putting $j=0, 1, 2, \ldots$ in (4) we obtain

$$j=0 \qquad u_0=-2\phi_{,x}^2 \tag{5}$$

$$j=1 \qquad u_1=2\phi_{,xx} \tag{6}$$

$$j=2 \qquad \phi_{,x}\phi_{,t}+6u_2\phi_{,x}^2-3\phi_{,xx}^2+4\phi_{,x}\phi_{,xxx}=0 \tag{7}$$

$$j=3 \qquad \phi_{,xt}-6u_3\phi_{,x}^2+6u_2\phi_{,xx}+\phi_{,xxx}+\phi_{,x}f(t)=0 \tag{8}$$

$$j=4 \qquad \frac{\partial}{\partial x}(\phi_{,xt}-6u_3\phi_{,x}^2+6u_2\phi_{,xx}+\phi_{,xxx}+\phi_{,x}f(t)) \tag{9}$$

where (9) is the compatibility condition. When we assign $u_4=u_6=0$ and $u_3=0$, we obtain

$$u_j=0 \qquad \text{for all } j\geqslant 3 \tag{10}$$

provided $u_2(x, t)$ is a solution of the PKdV equation,

$$u_{2,t}+6u_2u_{2,x}+u_{2,xxx}+f(t)u_2=0. \tag{11}$$

From (5)-(11) we obtain

$$u(x, t)=-2\phi_{,x}^2\phi^{-1}+2\phi_{,xx}\phi^{-1}+u_2 \tag{12}$$

and

$$\phi_{,x}\phi_{,t}+6u_2\phi_{,x}^2-3\phi_{,xx}^2+4\phi_{,x}\phi_{,xxx}=0 \tag{13}$$

$$\phi_{,xt}+6u_2\phi_{,xx}+\phi_{,xxx}+f(t)\phi_{,x}=0. \tag{14}$$

The Lax pair is obtained from (13) and (14) by using the transformation (Chen 1976)

$$\phi_{,x}=V^2. \tag{15}$$

Inserting (15) in (13) yields

$$V_{,t}+6u_{2,x}V^2+6u_2V_{,x}+4V_{,xxx}=0 \tag{16}$$

and (14) gives

$$V_{,t}+6u_2V_{,x}+3V_{,x}V_{,xx}V^{-1}+V_{,xxx}+\tfrac{1}{2}Vf(t)=0. \tag{17}$$

On eliminating $V_{,t}$ we obtain

$$3(V_{,xx}V^{-1})_{,x}+3u_{2,x}-\tfrac{1}{2}f(t)=0. \tag{18}$$

Integration yields

$$(D_x^2+u_2-\tfrac{1}{6}xf(t))V=\tfrac{1}{3}\mu V \tag{19}$$

where μ is an integration constant and $D_x^2=\partial^2/\partial x^2$. Let

$$g(t)(D_x^2+u_2-\tfrac{1}{6}xf(t))V=\lambda V \tag{20}$$

where

$$\lambda=\tfrac{1}{3}\mu g(t) \tag{21}$$

or

$$LV=\lambda V \tag{22}$$

where

$$L = g(t)(D_x^2 + u_2 - \tfrac{1}{6}xf(t)). \tag{23}$$

Also, (17) gives

$$V_{,t} = -MV \tag{24}$$

where

$$M = (4D_x^3 + 6u_2 D_x + 3u_{2,x}). \tag{25}$$

Using the consistency condition

$$L_{,t} = LM - ML \tag{26}$$

we obtain

$$f(t) = 1/(C_0 + 2t) \tag{27}$$

and

$$g(t) = C_1(C_0 + 2t) \qquad C_1 \neq 0 \tag{28}$$

where C_1 and C_0 are arbitrary integration constants. Hence the admissible Lax pair for the PKdV equation (1) when $f(t)$ is of the form of (27) is given by

$$L = C_1(C_0 + 2t)[D_x^2 + u - x/6(C_0 + 2t)] \tag{29}$$

$$M = (4D_x^3 + 6uD_x + 3u_{,x}) \tag{30}$$

implies that the PKdV equation (1) belongs to the class of inverse scattering technic (IST) solvable equations. For the integrability of the dynamical system, one has to show the existence of an infinite set of commuting symmetries. For that, we will find recursion operator from the Lax pair, using a recently introduced method (Oevel and Fokas 1984, Stramp 1984).

The transformation

$$\phi(x, t) = D_x(\psi^2) \tag{31}$$

changes solution of

$$L\psi = \lambda\psi \tag{32}$$

$$M\psi = -\psi_{,t} \tag{33}$$

into the solution of the following equation:

$$A\phi = -D_x^3\phi - uD_x\phi - u_{,x}\phi \tag{34}$$

where $A\phi$ is the linearisation (Fuchssteiner 1983) of the PKdV equation (1). The inverse transformation

$$\psi^2 = D_x^{-1}(\phi) \tag{35}$$

of the transformation (31) changes (32) into

$$(C_0 + 2t)\left[D_x^2 + 4u - \frac{2x}{3(C_0 + 2t)} + \left(2u_{,x} - \frac{1}{3(C_0 + 2t)}\right)D_x^{-1}\right]\phi = \lambda\phi/C_1 \tag{36}$$

implies that the recursion operator $R(u, x, t)$ for the PKdV equation (1) is given by

$$R(u, x, t) = (C_0 + 2t)\left[D_x^2 + 4u - \frac{2x}{3(C_0 + 2t)} + \left(2u_{,x} - \frac{1}{3(C_0 + 2t)}\right)D_x^{-1}\right]. \tag{37}$$

Using (34) and (37) we obtain

$$[A, R] = -(C_0+2t)\left(4u_3 - 2D_x^2/(C_0+2t) + 2u_4 D_x^{-1} + 24uu_{,x} + 12uu_{,xx}D_x^{-1}\right.$$
$$\left. + 12u_{,x}^2 D_x^{-1} - 4u/(C_0+2t) - \frac{2u, xD_x^{-1}}{(C_0+2t)}\right) \qquad (38)$$

where [,] is the commutator.

$$[D_t, R] = 2(D_x^2+4u) + (C_0+2t)(4u_{,t} + 2u_{,xt}D_x^{-1}) + 4u_{,x}D_x^{-1} \qquad (39)$$

implies

$$[A - D_t, R] = 0 \qquad (40)$$

where $D_t = \partial/\partial t$.

According to Olver (1986) the above condition (equation (40)) is a sufficient condition for the existence of an infinite number of commuting symmetries. Hence the PKdV equation (1) with $f(t)$ as in equation (27) is a completely integrable system and its auto-Bäcklund transformation is given by (11)-(14).

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